Nonlinear Robust Control Design for a 6 DOF Parallel Robot

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A class of robust tracking controllers for a 6 DOF parallel robot in the presence of nonlinearites and uncertainties are proposed. The controls are based on Lyapunov approach and guarantees practical stability. The controls utilize the information of link displacements and its velocities rather than using the positions or angles of the 6 DOF platform. This can be done by constructing the linkspace coordinates and the workspace coordinates simultaneously by imposing geometric constraints. The controls utilize the possible bound of uncertainty, and the uniform ultimate ball size can be adjusted by a suitable choice of control parameters. The control performance of the proposed algorithms is verified through experiments.

Key Words: Robust Tracking Control, Stewart Platform, Lyapunov Approach, Practical Stability, Uncertainty

Nomenclature

A : Hurwitz matrix
$\bar{B}$, ΔB : Nominal and uncertain input matrix
B_i, $i=1, 2, \cdots, 6$: Base joint vector
d_z, d_{z1} : Uniform bound ball in control system and modified system
D : Translational vector
e : Tracking error
$h(\cdot)$, $E(\cdot)$: Matching function in state and input
J : Jacobian matrix
$K=[K_p\ K_v]$, K_{p1},K_{v1} : Control gain in original system and modified system
l_i : i-th link length
$M(\cdot)$, $C(\cdot)$, $G(\cdot)$: Inertia, Coriolis, gravitational matrix or vector
$M_1(\cdot)$, $C_1(\cdot)$, $G_1(\cdot)$: Modified inertia, Coriolis, gravitational matrix or vector
$\bar{M}_1$, ΔM_1 : Nominal and uncertain inertia matrix
p_1 : Control term compensating uncertainty
P : Positive definite matrix
P_i^P, $i=1, 2, \cdots, 6$: Platform joint vector
$q=[u\ v\ w\ \alpha\ \beta\ \gamma]^T$: 6 DOF displacement
Q : Positive semidefinite matrix
$R_{\alpha\beta\gamma}$: Rotational matrix
R_z, R_{z1} : Bounds in control system and modified system
S_1 : Weighting in control
T_z, T_{z1} : Reaching time to uniform ultimate bound in control system and modified system
u : Control input
$\dot{y}^d$, $\ddot{y}_d$: Desired link velocity and acceleration
V : Lyapunov function
z : State variable

Greek characters

δ_ε : Uniform stability bound
$\lambda_E(q)$: Input uncertainty bound

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$\bar{\varepsilon}$: Control gain
η, η_1 : Coefficients in the 2nd order terms in original and modified system
$\mu(\cdot)$: Function utilized in control design
$\rho(\cdot)$, $\rho_1(\cdot)$: Bounding functions in original and modified system
$\bar{\sigma}\cdot\underline{\sigma}$: Upper and lower bound of inertia matrix
τ_1, τ_2 : Minimum and maximum of $\underline{\Omega}_{1i}$, $\bar{\Omega}_{1i}$
$\phi(\cdot)$, $\phi_1(\cdot)$: Uncertain functions in original and modified system
ξ : State variable
$\underline{\Omega}_{1i}$, $\bar{\Omega}_{1i}$: Lower and upper matrices in Lyapunov function

Superscript

p : Platform
T : Transpose of matrix
d : Desired value
-1 : Inverse matrix

Subscript

i : Link index
p, v : Position and velocity gain
z : State variable
1, 2, 3 : Class function on Lyapunov function

1. Introduction

A Stewart platform is a parallel manipulator system with a high force-to-weight ratio compared with conventional serial manipulators. Since serial manipulators generally have long reach and large workspace, they have low stiffness and other undesired characteristics, especially at a high speeds and heavy payloads due to the flexible structure. Since the appearance of the Stewart platform, many researchers have paid tremendous attention to it. The mechanism has been applied to flight simulators, robot manipulators, robot en d-effectors, and machine-tools. Many research activities have been devoted to kinematic and dynamic problems, *i.e.*, forward kinematics, dynamics including legs, and manipulator design. As for the control aspects, classical PID control has been employed in real applications even if it only gives mediocre control performance. However, high nonlinearity and uncertainty prevent a control algorithm from being developed compared to serial manipulators. For a 2 DOF parallel manipulator a control scheme has been proposed (Nguyen et al., 1986) and the tracking control has been reported for the Stewart platform system (Lebret et al., 1993). These controls rely on the exact knowledge of parameters. In real situations, the payload and parameters may be unknown (uncertain); thus it is difficult to design an appropriate controller a counting for the uncertainty. Adaptive control schemes whose controller gains are regulated by an adaptation law (Nguyen et. al., 1993) is one of the approaches to solve this problem. The control mainly dedicates to a time invariant system or a system with slowly time-varying parameters. As another alternative, robust control potentially offers a means of tackling the time-varying uncertain system. As for serial robots, several robust controls have been reported (Dawson et al., 1990)-(Qu, 1993).

In this article, we propose a class of robust control schemes for a Stewart platform which has a parallel structure. A conventional control which is shown in Fig. 1 is a type of tracking control for following the desired link lengths computed from

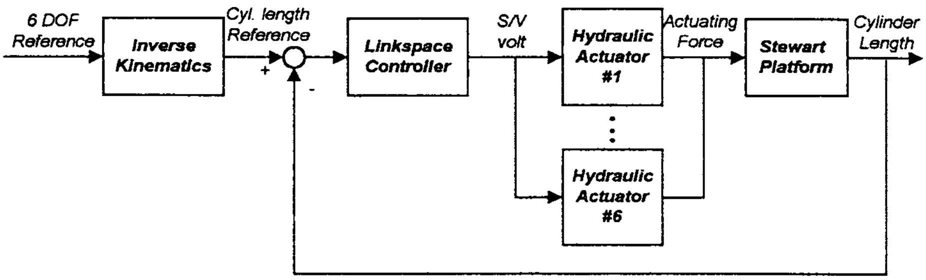

Fig. 1 Block diagram of control based on linkspace coordinates.

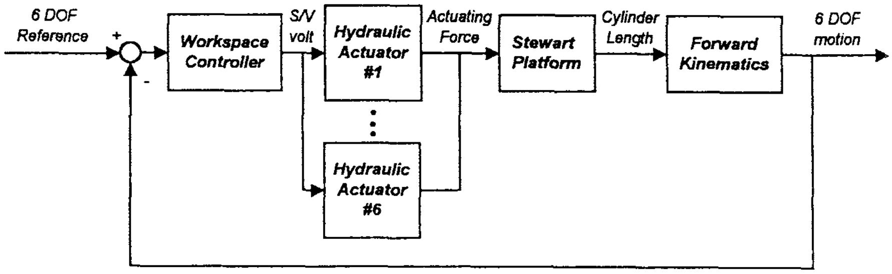

Fig. 2 Block diagram of control based on workspace coordinates.

the position command of the platform by inverse kinematics, which is called linkspace coordinate control. Most controllers in applications are based on linkspace coordinates (Nquyen et. al., 1993; Begon et al., 1995), which consider only an approximated manipulator model. Another control scheme uses the information of the top plate in control design (Kang et al., 1996). The control utilizes the dynamics which is similar to a serial manipulator, which is called workspace coordinate control (Fig. 2). However, the control based on workspace coordinates needs information from a 6 DOF sensor to measure the displacement or velocity if necessary. Or, it needs the forward kinematics solution to estimate the 6 DOF information which is based on the numerical method. To tackle these difficulties a robust control scheme based on linkspace coordinates is proposed in this paper together with a demonstration that the control guarantees practical stability (Chen, 1996).

2. Kinematics and Dynamics of a Stewart Platform

The coordinates reguired to represent a 6 DOF motion are given in terms of an inertial frame and the body-fixed frame attached to the moving platform. The 6 DOF motions consist of combined linear and angular motions. Linear motions consist of longitudinal (surge), lateral (sway), and vertical (heave) motions. Angular motions are described by Euler angles whose rotational sequences are x-axis, y-axis and z-axis. Here, we denote q as the 6 DOF displacement vector with elements surge(u), sway(v), heave(w), roll(α), pitch(β), and yaw(γ).

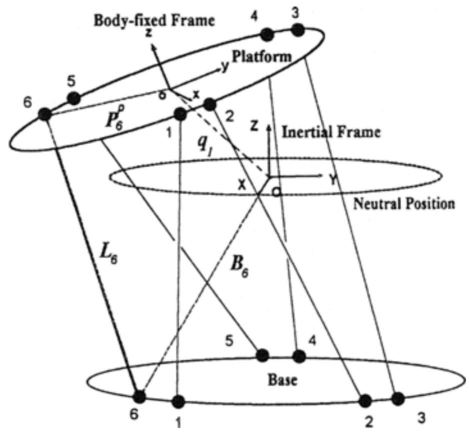

Fig. 3 Coordinates of Stewart platform.

$$q=[u\ v\ w\ \alpha\ \beta\ \gamma]^T. \tag{1}$$

In this article, we use the following notations in the model of the Stewart Platform. Referring to Fig. 3, we fixed an inertial frame (OXYZ) at the base platform, and a body-fixed frame ($oxyz$) at the top platform. The 12 joint coordinates are denoted as follows:

P_i^P, $i=1, 2, \cdots, 6$: a platform joint vector in the body-fixed frame.

B_i, $i=1, 2, \cdots, 6$: a base joint vector in the inertial frame.

If the rotational transformation matrix and the translation vector are represented by $R_{\alpha\beta\gamma}$ and D, respectively, the relative vector of the i th joint is written as

$$l_i=R_{\alpha\beta\gamma}P_i^P+D-B_i. \tag{2}$$

Thus, we can compute the link lengths, *i. e.*, the norm of l_i, from the given position and orientation of the platform. This problem is called the *inverse kinematic problem* of a Stewart platform.

The *forward kinematic problem* is the reverse of the inverse kinematics, *i. e.*, to get the position and orientation from the given actuator lengths. Because the solution of the forward kinematic problem can be analytically represented as the roots of a 16th or 40th order polynomial (Nair and Maddocks, 1994) polynomial roots are not easy to be solved. Thus, we usually use a numerical solution such as the Newton-Rhapson method (Nguyen et. al., 1993) in order to solve the forward kinematics problem.

Next, we introduce the dynamic model of a Stewart platform, which neglects the inertial motion of the links.

$$M(q, \sigma)\ddot{q} + C(q, \dot{q}, \sigma)\dot{q} + G(q, \sigma) = J^T(q)u. \tag{3}$$

Here q represents the displacement vector as shown in Eq. (1). $M(\cdot)$ is the inertia matrix, $C(\cdot)$ is the Coriolis and centrifugal force, $G(\cdot)$ is the gravitational force, $J(\cdot)$ is the Jacobian matrix and $u \in R^6$ is the actuator force and torque at each actuator. $\sigma(\cdot)$ (constant or time-varying) denotes the uncertain parameter vector. The detailed elements of $M(\cdot)$, $C(\cdot)$, $G(\cdot)$, and $J(\cdot)$ are given in the Appendix.

The main issue of this article is to design a controller to guarantee high control performance in the presence of uncertainty. Here, we list the assumptions regarding the uncertainty.

Assumption 1. The uncertain parameter vector is such that $\sigma \in \Sigma \subset R^0$ where Σ is prescribed and compact.

At first, for stability analysis we introduce the concept of practical stability (Chen, 1996). We consider the following class of uncertain dynamical systems:

$$\dot{\xi}(t) = f(\xi(t), \sigma(t), t), \ \xi_0 = \xi(t_0), \tag{4}$$

where $t \in R$ is the time, $\xi(t) \in R^n$ is the state, $\sigma(t) \in R^0$ is the uncertainty, and $f(\xi(t), \sigma(t), t)$ is the system vector.

Definition 1. The uncertain dynamical system Eq. (4) is *practically stable* iff there exists a constant $\underline{d}_\xi > 0$ such that for any initial time $t_0 \in R$ and any initial state $\xi_0 \in R^n$, the following properties hold.

(i) *Existence and continuation of solutions*: Given $(\xi_0, t_0) \in R^n \times R$, system Eq. (4) possesses a solution $\xi(\cdot): [t_0, t_1) \to R^n$, $\xi(t_0) = \xi_0$, $t_1 > t_0$. Furthermore, every solution $\xi(\cdot): [t_0, t_1) \to R^n$ can be continued over $[t_0, \infty)$.

(ii) *Uniform boundedness*: Given any constant $\gamma_\xi > 0$ and any solution $\xi(\cdot): [t_0, \infty) \to R^n$, $\xi(t_0) = \xi_0$ of Eq. (4) with $\|\xi_0\| \le r_\xi$, there exists $d_\xi(r_\xi) > 0$ such that $\|\xi(t)\| \le d_\xi(r_\xi)$ for all $t \in [t_0, \infty)$.

(iii) *Uniform ultimate boundedness*: Given any constant $\bar{d}_\xi > \underline{d}_\xi$ and any $r_\xi \in [0, \infty)$, there exists a finite time $T_\xi(\bar{d}_\xi, r_\xi)$ such that $\|\xi_0\| \le r_\xi$ implies $\|\xi(t)\| \le \bar{d}_\xi$ for all $t \ge t_0 + T_\xi(\bar{d}_\xi, r_\xi)$.

(iv) *Uniform stability*: Given any $\bar{d}_\xi > \underline{d}_\xi$, there exists a $\delta_\xi(\bar{d}_\xi) > 0$ such that $\|\xi_0\| \le \delta_\xi(\bar{d}_\xi)$ implies $\|\xi(t)\| \le \bar{d}_\xi$ for all $t \ge t_0$.

In this paper, the norm is Euclidean and the matrix norm is an induced norm. Thus, $\|\Pi\|^2 = \lambda_{\max}(\Pi^T\Pi)$, where Π is a real matrix. $\lambda_{\min(\max)}(\Pi)$ stands for the min(max) eigenvalue of the designated matrix Π.

3. Robust Control Based on Linkspace Coordinates

The workspace coordinate control relies on both displacement and velocity information of the platform. To obtain these we need to compute the forward kinematics or install 6 DOF sensors mounted on the platform, which requires high cost as mentioned in the introduction. In the case of Stewart Platform-type manipulators, we are not able to adopt the analytic solution of the forward kinematics for implementation in real time applications. Therefore, we instead rely on numerical method even if it does not guarantee the exact value of the platform information. Also, the forward kinematics solution sometimes requires much computational effort. Hence, the necessity of the control scheme based on information like link length and velocity naturally arises. We consider a control scheme designed in link-space coordinates. Here, we propose a different control scheme utilizing the link information rather than the platform information. we try to modify the dynamic equations based on workspace coordinates to be fit into link space coordinates.

The new dynamic equation in linkspace starts from the following property by using the Jacobian matrix $J(\cdot)$.

$$\dot{q} = J^{-1}(q)\dot{y}, \tag{5}$$

where $\dot{y} \in R^6$ is the velocity vector of the 6 links. Then, we construct a new dynamic equation in linkspace coordinates.

$$M_1(q, \sigma)\ddot{y} + C_1(q, \dot{q}, \sigma)\dot{y} + G_1(q, \sigma) = u, \tag{6}$$

where

$$M_1(q, \sigma) = J^{-T}(q) M(q, \sigma) J^{-1}(q), \tag{7}$$
$$C_1(q, \dot{q}, \sigma) = J^{-T}(q) M(q, \sigma) J^{-1}(q) + J^{-T}(q) C(q, \dot{q}, \sigma) J^{-1}(q), \tag{8}$$
$$G_1(q, \sigma) = J^{-T}(q) G(q, \sigma). \tag{9}$$

Here, $y \in R^6$ represents the displacement vector. y and $\dot{y}$ can be measured by a feasible linear sensor with ease.

Define a tracking error $e \in R^6$ and its derivative $\dot{e} \in R^6$ in the sense of the actuator, i.e.,

$$e = y - y^d, \ \dot{e} = \dot{y} - \dot{y}^d, \tag{10}$$

where y^d and $\dot{y}^d$ represent the desired actuator (link) displacement and velocity, respectively. Then, the error dynamic equation is given as follows:

$$M_1(q, \sigma)\ddot{e} + C_1(q, \dot{q}, \sigma)\dot{e} = -M_1(q, \sigma)\ddot{y}^d - C_1(q, q, \sigma)\dot{y}^d - G_1(q, \sigma) + u. \tag{11}$$

Here, $M_1(q, \sigma)$, $C_1(q, \dot{q}, \sigma)$, and $G_1(q, \sigma)$ are not necessarily expressed in terms of actuator variables y and $\dot{y}$. This is since the subsequent control scheme is able to handle the platform information q and $\dot{q}$. We express $M_1(\cdot)$ as the sum of the nominal value, which is only dependent on the known parameters and is computed from the neutral position, and the unknown term as

$$M_1^{-1} = \bar{M}_1^{-1}(0, \bar{\sigma}) + \Delta M_1^{-1}(q, \sigma), \tag{12}$$

where $\bar{\sigma}$ represents the nominal value of the parameter vector. Then Eq. (11) is expressed as Eq. (13).

$$\ddot{e} = \bar{M}_1^{-1} u + \Delta M_1^{-1}(q, \sigma) u + \phi(q, \dot{q}, e, \dot{e}, \dot{y}^d, \ddot{y}^d, \sigma), \tag{13}$$

where

$$\phi(q, \dot{q}, e, \dot{e}, \dot{y}^d, \ddot{y}^d, \sigma) = -\ddot{y}^d - M_1^{-1}(q, \sigma) C_1(q, \dot{q}, \sigma)(\dot{e} + \dot{y}^d) - M_1^{-1}(q, \sigma) G_1(q, \sigma). \tag{14}$$

express Eq. (13) can be expusud in state space form as follows:

$$\dot{z} = Az + \bar{B}u + \Delta Bu + \Phi(e, \dot{e}, q, \dot{q}, \dot{y}^d, \ddot{y}^d, \sigma) \tag{15}$$

where

$$z = [e \ \ \dot{e}]^T, \ A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$
$$\bar{B}(q) = \begin{bmatrix} 0 \\ \bar{M}_1^{-1}(0, \bar{\sigma}) \end{bmatrix},$$
$$\Delta B(q, \sigma) = \begin{bmatrix} 0 \\ \Delta M_1^{-1}(q, \sigma) \end{bmatrix} \tag{16}$$
$$\Phi(\cdot) = \begin{bmatrix} 0 \\ \phi(\cdot) \end{bmatrix}$$

From Eq. (16) it can be seen that the matching condition is satisfied, *i. e.*, the following conditions hold:

$$\Phi(e, \dot{e}, q, \dot{q}, \dot{y}^d, \ddot{y}^d) = \bar{B}h(e, \dot{e}, q, \dot{q}, \dot{y}^d, \ddot{y}^d), \tag{17}$$
$$\Delta B(q, \sigma) = \bar{B}E(q, \sigma). \tag{18}$$

From the above matching conditions the function $h(\cdot) \in R^6$ and $E(\cdot) \in R^{6\times 6}$ can be written as

$$h(e, \dot{e}, q, \dot{q}, \dot{y}^d, \ddot{y}^d) = \bar{M}(0, \bar{\sigma})\phi(e, \dot{e}, q, \dot{q}, \dot{y}^d, \ddot{y}^d), \tag{19}$$
$$E(q, \sigma) = \bar{M}(0, \bar{\sigma})\Delta M^{-1}(q, \sigma). \tag{20}$$

Under Assumption 1, we can choose a function $\rho(\cdot): R^6 \times R^6 \to R_+$ such that for all $\sigma \in \Sigma$,

$$\|h(e, \dot{e}, q, \dot{q}, \dot{y}^d, \ddot{y}^d, \sigma)\| \le \rho(e, \dot{e}). \tag{21}$$

Here, we can choose the bounding function $\rho(\cdot)$ such that it has a dependency on e and $\dot{e}$ only. This is since $\dot{q}$ can be transformed into $\dot{e}$ by $J(q)$, whose norm can be bound by a constant value, and the platform displacement q can be bounded by a constant within the specified workspace.

For the next step in designing a robust control, we consider the condition on input uncertainty.

Assumption 2. There exists a $\lambda_E(q)$ for all $q \in R^6$ such that

$$\max_{\sigma\in\Sigma}\|E(q,\sigma)\|=:\lambda_E(q)<1. \quad (22)$$

This assumption implies how much the uncertainty varies over the nominal value and the nominal value $\bar{M}(0,\bar{\sigma})$ is not far from $\Delta M^{-1}(q,\sigma)$. If the ratio $\bar{M}(0,\bar{\sigma})$ to $\Delta M^{-1}(q,\sigma)$ is less than 1 the condition holds, in which case we can compute the value of $\lambda_E(q)$. This assumption also implies that the nominal value $\bar{M}(q)$ is not far from $\Delta M^{-1}(q,\sigma)$.

Next, let the function $\bar{\rho}(\cdot): R^6\times R^6\to R_+$ be chosen such that

$$\bar{\rho}(e,\dot{e})\geq\frac{1}{1-\lambda_E(q)}\rho(e,\dot{e}). \quad (23)$$

Now, we construct a robust control $u\in R^6$ as

$$u=Kz+p(e,\dot{e})=[K_p\ K_v]\begin{bmatrix}e\\ \dot{e}\end{bmatrix}+p(e,\dot{e}). \quad (24)$$

Here, $K\in R^{6\times12}$ whose elements are $K_p\in R^{6\times6}$ an $K_v\in R^{6\times6}$ is chosen such that $\bar{A}(=A-BK)$ is Hurwitz; hence a positive symmetric definite matrix P satisfies

$$P\bar{A}+\bar{A}P=-Q,\ Q>0. \quad (25)$$

Also, $p(\cdot)\in R^6$ and $\mu(\cdot)\in R^6$ are represented as

$$p(e,\dot{e})=\begin{cases}-\dfrac{\mu(e,\dot{e})}{\|\mu(e,\dot{e})\|}\bar{\rho}(e,\dot{e}) & \text{if } \|\mu(e,\dot{e})\|>\varepsilon,\\ -\dfrac{\mu(e,\dot{e})}{\varepsilon}\rho^{-2}(e,\dot{e}) & \text{if } \|\mu(e,\dot{e})\|\leq\varepsilon\end{cases} \quad (26)$$

$$\mu(e,\dot{e})=\bar{B}^TPz. \quad (27)$$

Here, the hydraulic forces or torques are the control inputs in this scheme (Fig. 4). The dynamics between the voltages of the servo-valve and the hydraulic forces (torques) can be negligible because the servo-valve dynamics is far faster than that of the hydraulic actuators. Therefore, we assume that the voltages of the servo-valves are proportional to the forces (torques) of the hydraulic actuators.

Theorem 1. Subject to Assumptions 1 and 2, the system Eq. (15) is practically stable under the control Eq. (24).

Proof.

Define a Lyapunov candidate V as the following:

$$V=\frac{1}{2}z^TPz \quad (28)$$

The time derivative of V along the trajectory of the system Eq. (15) is given as

$$\begin{aligned}\dot{V}&=z^TP\dot{z}\\ &=z^TP(Az+\bar{B}u+\Delta Bu+\Phi).\end{aligned} \quad (29)$$

From Eqs. (18), (25), and (24) it can be seen that

$$\begin{aligned}\dot{V}&\leq z^TP(\bar{A}z+\bar{B}p+\bar{B}Ep+\bar{B}h)+\|P\bar{B}EK\|\ \|z\|^2\\ &=-\frac{1}{2}z^TQz+z^TP\bar{B}(p+Ep+h)\\ &\leq-(\frac{1}{2}\lambda_{\min}(Q)-\lambda_E\|P\bar{B}K\|)\ \|z\|^2\\ &\quad+z^TP\bar{B}(p+Ep+h).\end{aligned} \quad (30)$$

When $\|\mu\|>\varepsilon$, the last second term of Eq. (30) follows from Eq. (26)

$$\begin{aligned}&z^TP\bar{B}(p+Ep+h)\\ &=-\frac{\|z^TP\bar{B}\|^2}{\|\bar{B}^TPz\|}\bar{\rho}+\|z^TP\bar{B}\|\ \|E\|\bar{\rho}+\|z^TP\bar{B}\|\rho\\ &\leq-\|z^TP\bar{B}\|\bar{\rho}+\|z^TP\bar{B}\|(\lambda_E\bar{\rho}+\rho)\\ &\leq\|z^TP\bar{B}\|(-\bar{\rho}+\lambda_E\bar{\rho}+\rho)\leq0.\end{aligned} \quad (31)$$

When $\|\mu\|<\varepsilon$, it follows from Eqs. (2) and (26)

$$\begin{aligned}&z^TP\bar{B}(p+Ep+h)\\ &=-\|z^TP\bar{B}\|^2\frac{\bar{\rho}^2}{\varepsilon}+\|z^TP\bar{B}\|\lambda_E\bar{\rho}+\|z^TP\bar{B}\|\rho\end{aligned}$$

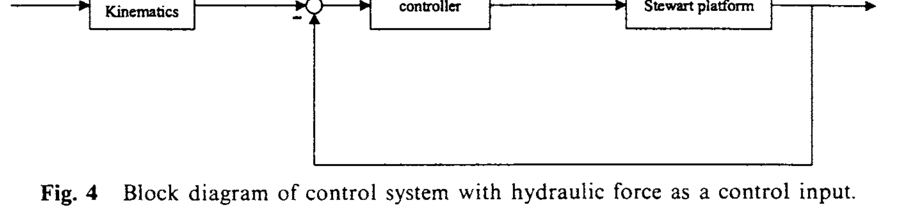

Fig. 4 Block diagram of control system with hydraulic force as a control input.

$$\le -\|z^T P\bar{B}\|^2 \frac{\bar{\rho}^2}{\varepsilon} + \|z^T P\bar{B}\| \bar{\rho} \le \frac{\varepsilon}{4}. \quad (32)$$

Therefore, from Eqs. (31) and (32), $\dot{V}$ is bounded by

$$\dot{V} \le -\frac{1}{2} z^T Q z + \frac{\varepsilon}{4} \le -\eta \|z\|^2 + \bar{\varepsilon}, \quad (33)$$

where

$$\eta = \frac{1}{2}\lambda_{\min}(Q) - \lambda_E \|P\bar{B}K\|, \quad (34)$$

$$\bar{\varepsilon} = \frac{\varepsilon}{4}. \quad (35)$$

Therefore, $\dot{V} < 0$ for all $\|z\| > R_z$, where

$$R_z = \sqrt{\frac{\bar{\varepsilon}}{\eta}}. \quad (36)$$

Following Eq. (33) for $r_z \ge 0$, if $\|z_0\| \le r_z$, we can satisfy the requirements of uniform boundedness, uniform ultimate boundedness and uniform stability (Corless and Leitmann, 1981) by selecting

$$d_z(r_z) = \begin{cases} (\gamma_1^{-1} \circ \gamma_2)(R_z) & \text{if } r_z \le R_z \\ (\gamma_1^{-1} \gamma_2)(r_z) & \text{if } r_z > R_z, \end{cases} \quad (37)$$

$$T_z(\bar{d}_z, r_z) = \begin{cases} 0 & \text{if } r_z \le \bar{d}_z \sqrt{\frac{\gamma_1}{\gamma_2}} \\ \frac{\gamma_2(r_z) - (\gamma_1 \circ \gamma_2^{-1} \circ \gamma_1)(\bar{d})}{(\gamma_3 \circ \gamma_2^{-1} \circ \gamma_1)(d)} & \text{otherwise}, \end{cases} \quad (38)$$

$$\delta_z(\bar{d}_z) = R_z, \quad (39)$$

where $\gamma_1(z) = \frac{1}{2}\lambda_{\min}(P)\|z\|^2$, $\gamma_2(z) = \frac{1}{2}\lambda_{\max}(P)\|z\|^2$, $\underline{d}_z = \gamma_1^{-1} \circ \gamma_2(R_z)$, $\gamma_3(\|z\|) = \eta\|z\|^2$, $\bar{R}_z = \gamma_2^{-1} \circ \gamma_1(\bar{d}_z)$. Q. E. D.

Remark 1. The control in linkspace coordinates does not require the forward kinematics solution which computes the platform displacement from the link displacement. It means that the control does not need to devote much computational effort for the numerical solution, or that it is not necessary to install a 6 DOF motion sensor. However, computing the bounding function $\rho(\cdot)$ may give a conservative control because we assumed that $\rho(\cdot)$ is dependent on only e and $\dot{e}$ by assigning the bound of q as a constant as mentioned in Section 3.

4. Another Efficient Robust Control Design

Since the control proposed in Section Ⅲ utilizes the inverse matrix $M^{-1}(\cdot)$ in computing the bounding function $\rho(\cdot)$, it is sometimes troublesome in real time application. Here, an alternative control type is proposed instead. The control results from a geometry dependent Lyapunov function.

The following additional assumption is needed to design another robust control for the Stewart Platform system.

Assumption 3. There exist positive constants $\underline{\sigma}$ and $\bar{\sigma}$ such that

$$\underline{\sigma} I \le M_1(q, \sigma) \le \bar{\sigma} I, \ \forall q \in R^6, \ \forall \sigma \in \Sigma. \quad (40)$$

Now, given $S_1 = \text{diag}[S_{1i}]_{6\times6}$, $i = 1, 2, \cdots, 6$, $S_{1i} > 0$, and a scalar $\varepsilon_1 > 0$, we propose a robust control $u \in R^6$:

$$u = (-K_{p_1} e - K_{v_1} \dot{e} + p_1), \quad (41)$$

where

$$p_1(e, \dot{e}) \in R^6 = \begin{cases} -\frac{\mu_1(e, \dot{e})}{\|\mu_1(e, \dot{e})\|}\rho_1(e, \dot{e}) & \text{if } \|\mu_1(e, \dot{e})\| > \varepsilon_1 \\ -\frac{\mu_1(e, e)}{\varepsilon_1}\rho_1(e, \dot{e}) & \text{if } \|\mu_1(e, \dot{e})\| \le \varepsilon_1, \end{cases} \quad (42)$$

$$\mu_1(e, \dot{e}) = (e + S_1 \dot{e})\rho_1(e, \dot{e}) \in R^6, \quad (43)$$

$$K_{p_1} := \text{diag}[K_{p_{1i}}]_{6\times6}, \ K_{p_{1i}} > 0, \ i = 1, 2, \cdots, 6, \quad (44)$$

$$K_{v_1} := \text{diag}[K_{v_{1i}}]_{6\times6}, \ K_{v_{1i}} > 0. \ i = 1, 2, \cdots, 6, \quad (45)$$

Here, $\rho_1(\cdot): R^6 \times R^6 \to R_+$ is the bounding function computed from

$$\|\phi_1(q, \dot{q}, e, \dot{e}, \dot{q}^d, \ddot{q}^d, \sigma)\| \le \rho_1(e, \dot{e}), \quad (46)$$

where

$$\phi_1(\cdot) = -M_1(q, \sigma)(\ddot{q}^d - S_1 \dot{e}) - C_1(q, \dot{q}, \sigma)(\dot{q}^d - S_1 e) - G_1(q, \sigma). \quad (47)$$

Theorem 2. Subject to Assumptions 1, 2 and 3, the system Eq. (15) is practically stable under the control Eq. (41).

Proof.

Define a Lyapunov function candidate V_1 as the following:

$$V_1=\frac{1}{2}(\dot{e}+S_1e)^TM_1(\dot{e}+S_1e)$$
$$+\frac{1}{2}e^T(K_{p_1}+S_1K_{v_1})e. \quad (48)$$

To see that V_1 is a legitimate Lyapunov function candidate, we shall prove that V_1 is positive definite and decrescent. By Assumption 2, it can be seen that

$$V_1\geq\frac{1}{2}\underline{\sigma}\|\dot{e}+S_1e\|^2+\frac{1}{2}e^T(K_{p_1}+S_1K_{v_1})e$$
$$=\frac{1}{2}\underline{\sigma}\sum_{i=1}^{6}(\dot{e}_{1_i}{}^2+2S_{1_i}\dot{e}_{1_i}e_{1_i}+S_{1i}^2\ e_{1_i}^2)+\frac{1}{2}\sum_{i=1}^{6}(K_{p_{1i}}$$
$$+S_{1_i}K_{v_{1_i}})e_{1_i}^2=\frac{1}{2}\sum_{i=1}^{6}[e_{1_i}\dot{e}_{1_i}]\underline{\Omega}_{1_i}\begin{bmatrix}e_{1_i}\\ \dot{e}_{1_i}\end{bmatrix}, \quad (49)$$

where

$$\underline{\Omega}_{1_i}:=\begin{bmatrix}\underline{\sigma}S_{1i}^2+K_{p_{1i}}+S_{1_i}K_{v_{1i}} & \underline{\sigma}S_{1i}\\ \underline{\sigma}S_{1i} & \underline{\sigma}\end{bmatrix}. \quad (50)$$

where e_{1i} and $\dot{e}_{1i}$ are the ith components of e and $\dot{e}$, respectively. For the sake of avoidig confusion, we introduce a state variable $z_1=[e\ \ \dot{e}]^T$ different from the z defined in Eq. (16).

Since $\underline{\Omega}_{1i}>0$, $\forall i$, V_1 is positive definite:

$$V_1\geq\frac{1}{2}\sum_{i=1}^{6}\lambda_{\min}(\underline{\Omega}_{1i})(e_{1i}^2+\dot{e}_{1i}^2)\geq\tau_1\|z_1\|^2, \quad (51)$$

where

$$\tau_1:=\frac{1}{2}\min_{i}\left\{\min_{\sigma\in\Sigma}\lambda_{\min}(\underline{\Omega}_{1i}),\ i=1,2,\cdots,6\right\}. \quad (52)$$

Next, with respect to the bounded from the above condition in Assumption 2,

$$V_1\leq\frac{1}{2}\bar{\sigma}\|\dot{e}+S_1e\|^2+\frac{1}{2}e^T(K_{p_1}+S_1K_{v_1})e$$
$$=\frac{1}{2}\bar{\sigma}\sum_{i=1}^{6}(\dot{e}_{1_i}{}^2+2S_{1_i}\dot{e}_{1_i}e_{1_i}+S_{1i}^2\ e_{1_i}^2)$$
$$+\frac{1}{2}\sum_{i=1}^{6}(K_{p_{1_i}}+S_{1_i}K_{v_{1_i}})e_{1_i}^2$$
$$=\frac{1}{2}\sum_{i=1}^{6}[e_{1_i}\dot{e}_{1_i}]\bar{\Omega}_{1_i}\begin{bmatrix}e_{1_i}\\ \dot{e}_{1_i}\end{bmatrix}, \quad (53)$$

where

$$\bar{\Omega}_{1_i}:=\begin{bmatrix}\bar{\sigma}S_{1i}^2+K_{p_{1i}}+S_{1_i}K_{v_{1i}} & \bar{\sigma}S_{1i}\\ \bar{\sigma}S_{1i} & \bar{\sigma}\end{bmatrix}, \quad (54)$$

Therefore, we have

$$V_1\leq\frac{1}{2}\sum_{i=1}^{6}\lambda_{\max}(\bar{\Omega}_{1i})(e_{1i}^2+\dot{e}_{1i}^2)\leq\tau_2\|z_1\|^2, \quad (55)$$

where

$$\tau_2:=\frac{1}{2}\max_{i}\left\{\max_{\sigma\in\Sigma}\lambda_{\max}(\bar{\Omega}_{1i}),\ i=1,2,\cdots,6\right\}. \quad (56)$$

The derivative of V_1 along the trajectory of the Eq. (11) is given by

$$\dot{V}_1=(\dot{e}+S_1e)^TM_1(\ddot{e}+S_1\dot{e})$$
$$+\frac{1}{2}(\dot{e}+S_1e)^T\dot{M}_1(\dot{e}+S_1e)+e^T(K_{p_1}+S_1K_{v_1})\dot{e}$$
$$=(\dot{e}+S_1e)^T(M_1S_1\dot{e}-M_1\ddot{q}^d-C_1\dot{q}^d-C_1\dot{e}$$
$$-G_1+u)+e^T(K_{p_1}+S_1K_{v_1})e$$
$$+\frac{1}{2}(\dot{e}+S_1e)^T\dot{M}_1(\dot{e}+S_1e). \quad (57)$$

According to Eqs. (41) and (47) and the skew-symmetric property of $\dot{M}_1-2C_1$ (Zribi and Ahmad, 1992), it can be seen that

$$\dot{V}_1=(\dot{e}+S_1e)^T(M_1S_1\dot{e}-M_1\ddot{q}^d-C_1\dot{q}^d$$
$$+C_{1S_1}e-G_1+p_1)-e^TS_1K_{p_1}e-\dot{e}^TK_{v_1}\dot{e}$$
$$\leq(\dot{e}+S_1e)^T\phi_1+(\dot{e}-S_1e)^Tp_1$$
$$-\lambda_{\min}(S_1K_{p_1},K_{v_1})(\|e\|^2+\|\dot{e}\|^2). \quad (58)$$

If $\|\mu_1\|>\varepsilon_1$, the first two terms in Eq. (58) become

$$(\dot{e}+S_1e)^T\phi_1+(\dot{e}+S_1e)^Tp_1\leq\|\dot{e}+S_1e\|\rho_1$$
$$+(\dot{e}+S_1e)^T\left(-\frac{\dot{e}+S_1e}{\|\dot{e}+S_1e\|}\right)\rho_1=0. \quad (59)$$

If $\|\mu_1\|\leq\varepsilon_1$, those become

$$(\dot{e}+S_1e)^T\phi_1+(\dot{e}+S_1e)^Tp_1$$
$$\leq\|\dot{e}+S_1e\|\rho_1+(\dot{e}+S_1e)^T\left(-\frac{\dot{e}+S_1e}{\varepsilon_1}\right)\rho_1^2$$
$$=\|\dot{e}_1+S_1E_1\|\rho_1-\frac{1}{\varepsilon_1}\|\dot{e}_1+S_1e_1\|^2\rho_1^2\leq\frac{\varepsilon_1}{4}. \quad (60)$$

Therefore, $\dot{V}_1$ is bounded by

$$\dot{V}_1\leq-\lambda_{\min}(S_1K_{p_1},K_{v_1})(\|e\|^2+\|\dot{e}\|^2)$$
$$+\frac{\varepsilon_1}{4}=:\ -\eta_1\|z_1\|^2+\bar{\varepsilon}_1, \quad (61)$$

where

$$\eta_1=\lambda_{\min}(S_1K_{k_1},K_{v_1}), \quad (62)$$
$$\bar{\varepsilon}_1=\frac{\varepsilon_1}{4}. \quad (63)$$

Thus, $\dot{V}_1<0$ for all $\|z_1\|>R_{z_1}$ where

$$R_{z_1}=\sqrt{\frac{\bar{\varepsilon}_1}{\eta_1}}. \quad (64)$$

Following Eq. (61) for $r_{z_1}\geq0$, if $\|z_1\|\leq r_{z_1}$, we can satisfy the requirements of uniform boundedness, uniform ultimate boundedness and uniform stability (Corless and Leitmann, 1981) by selecting

$$d_{z_1}(r_{z_1}) = \begin{cases} (\gamma_1^{-1}\circ\gamma_2)(R_{z_1}) & \text{if } r_{z_1} \le R_{z_1} \\ (\gamma_1^{-1}\gamma_2)(r_{z_1}) & \text{if } r_{z_1} > R_{z_1}, \end{cases} \quad (65)$$

$$T_{z_1}(\bar{d}_{z_1}, r_{z_1}) = \begin{cases} 0 & \text{if } r_{z_1} \le \bar{d}_{z_1}\sqrt{\frac{\gamma_1}{\gamma_2}} \\ \frac{\gamma_2(r_{z_1}) - (\gamma_1\circ\gamma_2^{-1}\circ\gamma_1(\bar{d})}{(\gamma_3\circ\gamma_2^{-1}\circ\gamma_1)(\bar{d})} & \text{oterwise,} \end{cases} \quad (66)$$

$$\delta_{z_1}(\bar{d}_{z_1}) = R_{z_1}, \quad (67)$$

where $\gamma_1(z_1) = \tau_1\|z_1\|^2$, $\gamma_2(z_1) = \tau_2\|z_1\|^2$, $\gamma_3(\|z_1\|) = \eta_1\|z_1\|^2$, $\underline{d}_z = \gamma_1^{-1}\circ\gamma_2(R_{z_1})$, $\bar{R}_{z_1} = \gamma_2^{-1}\circ\gamma_1(\bar{d}_{z_1})$. Q.E..

Remark 2. The controller shown above needs an assumption which requires that $M_1(\cdot)$ is lower and upper bound by constants. This condition is sometimes restrictive in applying to general robots. For instance, a SCARA type robot does not satisfy this condition. For the Stewart Platform the condition is satisfied. However, the control provides an efficient control by excluding the burden which computes the inverse matrix $M^{-1}(\cdot)$. The proposed control relies on the possible bound of uncertainty. Also, the control guarantees practical stability. The tracking error can be adjusted by a suitable choice of control parameter ε_1.

5. Experimental Validation

The control performance of the proposed control for the Stewart Platform is verified through experiments. The Stewart Platforms designed is shown in Fig. 5. The motion bed consists of 6 hydraulic cylinders attached to the top and bottom platform and 6 servo-valves to control the cylinders with each servo amplifier. Each cylinder is of a single rod type with a diameter of 75mm in the apper chamber, and 61.2 mm in the lower chamber. The full stroke of each cylinder is 0.4m. The nominal value of important parameters of those are given in Table 1. The servo valve is of a 30 series type with 30 l/min-rated flow rate and a time constant of 0.013 sec^{-1}. The hydraulic power pack used has a rated pressure phyaicity of 110 bar and a rated flow of 70 l/min.

To measure the cylinder displacements and velocities a LVDT (linear variable displacement transducers) is used. A 486 PC with 32 channel AD converter and 6 channel DA converter are used for control and data acquisition. The AD converter has 12-bit 20 channels (6 for cylinder

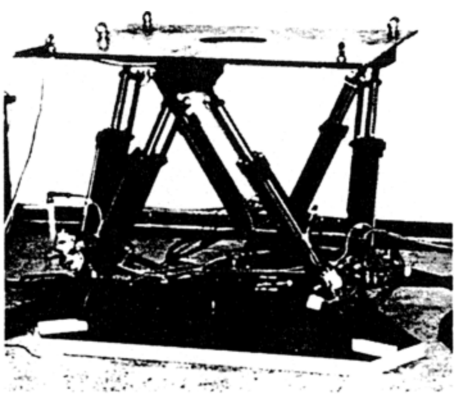

Fig. 5 6 DOF motion simulator for experiment.

Table 1 Feature parameters of Stewart platforms.

Variable	Description	Motion Bed	Prototype	Unit
u_m, v_m, w_m	maximum excursion of platform	0.640	0.320	m
$\alpha_m, \beta_m, \gamma_m$	maximum angle of platform	25	25	deg.
X_{max}	maximum stroke of hydraulic cylinder	±0.400	±0.200	m
m	mass of platform	2000	500	kg
$I_x, I_y(I_z)$	mass moment of inertia about x, y(z) axis	2000(4000)	500(1000)	kg · m^2
$R_p(R_b)$	radius of platform (base)	1.080(1.20)	0.48(0.60)	m
d_p, d_b	displacement of adjacent joints of platform and base	0.320	0.10	m

lengths, 14 for pressure monitoring), and the DA converter has 12-bit 6 channels to drive the servo amplifier for servo-valve operation. The unknown parameter values are estimated.

The linkspace controller gains $K_p = 1.5 * 10^6$, $K_v = 2 * 10^5$ are chosen and the control parameter $\varepsilon = 10^{-7}$ is chosen. linkspace robust control enables the system to follow the command signals in steady state within a satisfactory level.

Figures 6 and 7 illustrate the experimental results of tracking over sinusoidal command signals (rolling, pitching and yawing: $\sin(0.2 * 2\pi t) + 0.5\sin(0.6 * 2\pi t)$ deg.) with PD control and the robust control in linkspace presented in Section 3, respectively. The PD control performance results from carefultuning of the gains after several adjustments. In PD control the angles track the command signals with about 50% offset, whereas robust control provides less than 10% offset. Also, the phase lag can be reduced by 50% in robust control compared to PD control. We see

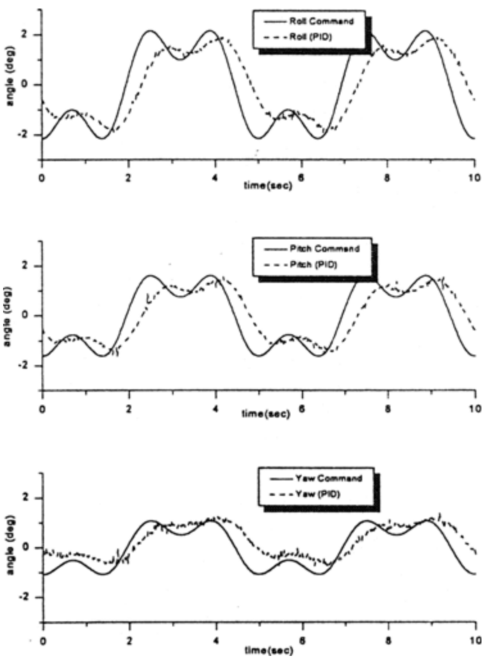

Fig. 6 Experimental results of the tracking histories of angular motions with PD control.

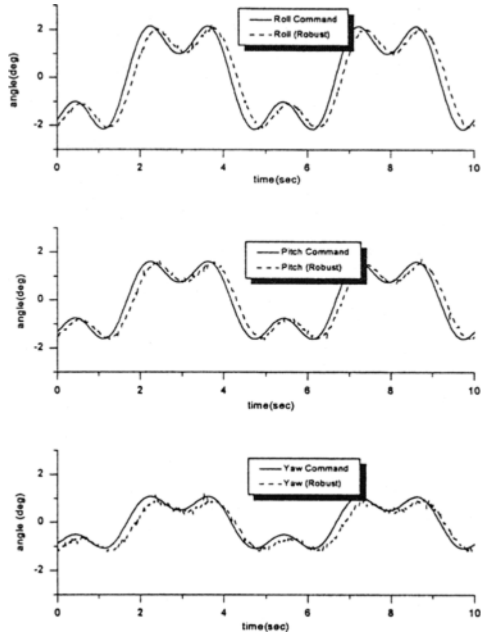

Fig. 7 Experimental results of the tracking histories of angular motions with linkspace robust control.

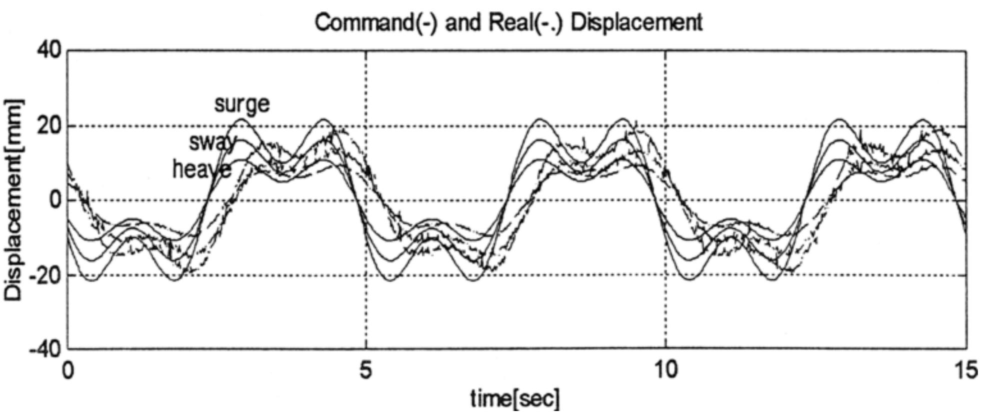

Fig. 8 Experimental results of the tracking histories of translational motions with PD control.

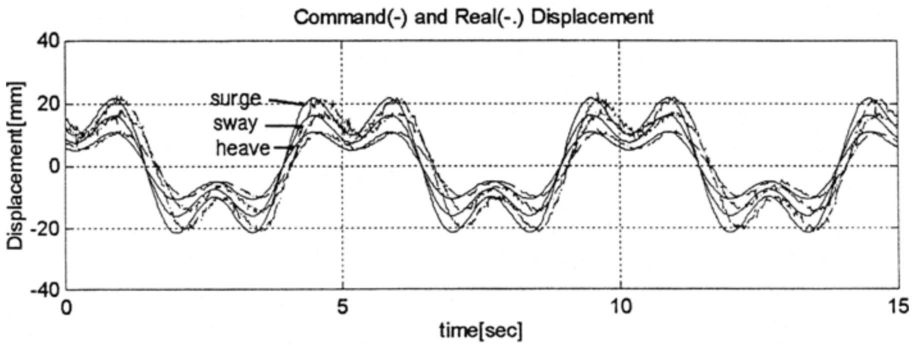

Fig. 9 Experimental results of the tracking histories of translational motions with linkspace robust control.

that robust control guarantees the three angular motions track the command signals properly. For translational motions the PD control has difficulty in tracking the command signals (Fig. 8) whereas robust control (Fig. 9) shows an enhanced tracking performance. The uniform ball can be increased due to the less value of η. I we decrease $\bar{\varepsilon}$ to get less R_z Chattering arises. Therefore, a compromise between tracking error and chattering is needed.

Viewing Fig. 7, the robust control does not show perfect tracking. The reason is mainly due to the effects of hydraulic components such as cylinders and servo valves. In other words, the hydraulic dynamics needs to be fully considered in the control design for better control performance. However, it shows much improved tracking performance compared with the PD control. We see that the proposed control tackles the uncertainty even if the payload varies with time or its value is not exactly known. The control only needs the bounds on uncertainty and adjusts the error bound by a control parameter ε. This fact is shown in Fig. 7 and 9.

6. Conclusion

A class of robust controls in the presence of uncertainty for a Stewart platform are proposed. The parallel manipulator system is similar in terms of mechanical function to a serial robot, but the control needs to be of a different form compared with that of a serial robot. The control schemes based on linkspace coordinates are introduced that tackle the uncertainty due to unknown geometric properties. Firstly, the control with a quadratic type Lyapunov function is presented. The control does not limit the upper and lower boundedness of the inertia matrix. However it requires the computation of the inverse of the inertia matrix. Secondly, a robust control based on a different Lyapunov function which depends on the boundedness of the manipulator, but gets rid of the inverse inertia matrix computation, is introduced to make the control scheme more efficient. The proposed controls handle the workspace coordinates and linkspace coordinates simultaneously in control designs. This is possible due to the use of the possible bound on a uncertainty and the geometric bound of the platform.

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Appendix

$$M(q)=\begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_xC_\beta^2C_\gamma^2+I_yC_\beta^2S_\gamma^2+I_zS_\beta^2 & (I_x-I_y)C_\beta C_\gamma S_\gamma & I_zS_\beta \\ 0 & 0 & 0 & (I_x-I_y)C_\beta C_\gamma S_\gamma & I_xS_\gamma^2+I_yC_\gamma^2 & 0 \\ 0 & 0 & 0 & I_zS_\beta & 0 & I_z \end{bmatrix}$$

$$C(q,\dot q)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_1\dot\beta+K_2\dot\gamma & K_1\dot\alpha+K_5\dot\beta+K_3\dot\gamma & K_2\dot\alpha+K_3\dot\beta \\ 0 & 0 & 0 & -K_1\dot\alpha+K_3\dot\gamma & K_4\dot\gamma & K_3\dot\alpha+K_4\dot\beta \\ 0 & 0 & 0 & -K_2\dot\alpha-K_3\dot\beta & -K_3\dot\alpha-K_4\dot\beta & 0 \end{bmatrix}$$

Jacobian J

$$J=J_1J_2$$

$$J_1=\begin{bmatrix} u_1^T & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} \\ 0_{1\times3} & u_2^T & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} \\ 0_{1\times3} & 0_{1\times3} & u_3^T & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & u_4^T & 0_{1\times3} & 0_{1\times3} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & u_5^T & 0_{1\times3} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & u_6^T \end{bmatrix}$$

J_2's $row=[I_{3\times3} \quad S(i)R_\alpha R_\beta R_\gamma P_i^P \quad R_\alpha S(j)R_\beta R_\gamma P_i^P \quad R_\alpha R_\beta S(k)R_\gamma P_i^P]$

where

$$u_i=\frac{R_{\alpha\beta\gamma}P_i+q_1-B_i}{\|R_{\alpha\beta\gamma}P_i+q_1-B_i\|},\quad R_{\alpha\beta\gamma}=R_\alpha R_\beta R_\gamma,$$

$$S(v)=\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix},\ \text{if } v=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$R_\alpha=\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\alpha & -S_\alpha \\ 0 & S_\alpha & C_\alpha \end{bmatrix},\ R_\beta=\begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix},$$

$$R_\gamma=\begin{bmatrix} C_\gamma & -S_\gamma & 0 \\ S_\gamma & C_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$G=[0\ 0\ -mg\ 0\ 0\ 0]^T.$$

$$K_1=-C_\beta S_\beta(C^2{}_\gamma I_x+S^2{}_\gamma I_y-I_z)$$

$$K_2=-C^2{}_\beta C_\gamma S_\gamma(I_x-I_y)$$

$$K_3=\frac{1}{2}C_\beta(C_\gamma-S_\gamma)(C_\gamma+S_\gamma)(I_x-I_y)$$

$$K_4=C_\gamma S_\gamma(I_x-I_y)$$

$$K_5=-C_\gamma S_\gamma S_\beta(I_x-I_y)$$